

ON THE MITTAG-LEFFLER-HYERS-ULAM STABILITY OF PRABHAKAR FRACTIONAL DIFFERENTIAL EQUATIONS THROUGH FRACTIONAL FOURIER TRANSFORMS

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Abstract This paper aims to study the stability of the linear and nonlinear fractional differential equations in Mittag-Leffler-Hyers-Ulam sense with Prabhakar derivative. By using the Fractional Fourier transform, we show that the introduced fractional differential equations are Mittag-Leffler Hyers-Ulam is stable with Prabhakar derivative. Some examples are given to illustrate the main theoretical results.

Keywords: Fractional differential equation, Riemann derivative, Prabhakar fractional integral, Prabhakar fractional derivative, Mittag-Leffler function, Fractional Fourier transform

1. Introduction

Fractional calculus is a generalization of ordinary calculus to non-integer differentiation and integration. There has been a lot of contributions to the theory of fractional calculus by famous Mathematicians are known: Laplace, Fourier, Abel, Leoville Letnikov, Heaviside, Weyl, Erdelyi, Go Renflo and Mainardi et al. Fractional differential equations have recently become a very strong tool in many fields such as Physics, Thermodynamics, Electrical circuit theory and fractalness, Seepage flow in porous media, Mechatronics systems, signal processing, Chemical mixing, Chaos theory, etc. Fractional differential equations and its applications are discussed up to date in books of Oldharm and Spanner Sankilbas, Marcher, Miller and Ross, Podlubny [1-5]. The most important advantage of using a fractional differential equation is their non-local property that is the next state of a system depend not only upon its current state but also upon all of its historical states. When describing differential, fractional and functional equations as physical models and practical problems, finding exact solutions to these equations is improbably difficult. If there exist exact solutions is often so complicated that is not convenient for numerical solutions. In view of this, it is necessary to discuss approximate solutions that lie near-exact solutions.

Stanislaw Marcin Ulam introduced the Hyers-Ulam stability problem in 1940. He motivated the study of the stability problems for various functional equations. He arises a list of unsolved problems with the stability of homomorphisms. In 1941 Hyer [6] gave a brilliant answer to Ulam's question the answer concerning the stability of functional equations for the case of approximately additive mapping G_1 and G_2 are assumed Banach spaces. This result plays a very important role in the study of the stability of several functional equations. Rassias proved the Hyers-Ulam stability for additive Cauchy equation later it is generated by Aoki for additive mappings [7,8]. Obbloza one of the first author who investigated the Hyers-Ulam stability of differential equations. The Mathematicians I.A. Rus, Takahashi, Wang Zhou, S.M. Jung, Aligifiary, Onitsuka et al [10-19] gave their contributions for Hyers Ulam stability in differential equations. Wang Zhou et al [20] proved the stability of fractional evolution equations and the stability of nonlinear differential equations with fractional integral impulses, and they also contributed some new concepts in the stability of the fractional differential equations. Liu et al [21] established the Hyers-Ulam stability of Linear Cauchy-Fabrizio fractional differential equations using Laplace transform method. J. Wang Y. Zhou [22] discussed Mittag-Leffler-Ulam stabilities of fractional evolution equations. V. Kalvandi et al proved Mittag-Hyers-Ulam stability of second-order differential equation and Emden of second order. Wang and Li et al obtained properties of Mittag-Leffler function and the Hyer-Ulam stability of fractional-order linear differential equation using Laplace transform method.

Swedish mathematician Magnus Gosta Mittag-Leffler defined a new special function for divergent series which is known as Mittag-Leffler (ML) function in 1948. Pollard [25] derived one parameter Mittag-Leffler function. Main features of ML function and generalization of two-parameter Mittag-Leffler function derived by P. Humbert, et al [26-28]. Extension of Mittag-Leffler function and the

solution of some linear differential equations of non-integer order discussed in [29,30]. K.S. Cole, B. Gross, M. Caputo, and F. Mandari gave their crucial contributions to fractional calculus and showing the Mittag-Leffler function always appears in the material functions of a system described by fractional-order equations. ML function was named the Queen function of the fractional calculus by F. Mandari and R. Gorenflo.

Special praise is due to the three-parameter extension of the ML function introduced by the Indian mathematician Tilak Raj Prabhakar in [31] dating back to 1971. Although this function was originally analytical within the framework of weakly singular Volterra integral equations from a purely mathematical perspective, it turns out that it plays an important role in the description of an anomalous dielectric relaxation. The effectiveness of the Prabhakar function is discussed in the Havriliak-Negami empirical law. In 2002 A. Kilbas, M. Saigo, and R.K. Saxena [32] developed Volterra operations involving the Prabhakar function. One of the left-inverse operators of the Prabhakar fractional integral, which will later become known as Prabhakar fractional derivatives (Riemann - Liouville type). M. D'Ovidio and F. Polito [33] provided a regularization of the fractional derivative and named it after Indian Mathematician R. Garra et al [34] where the features of the Prabhakar derivatives and some of their applications were discussed in depth. Relation between Prabhakar and Wright functions derived by L. Beghin and E. Orsingher [35]. R. B. Paris in the study of the Prabhakar function represents a special case, which has then led to a specific resolution on the Prabhakar function.

Stability region of fractional differential systems with Prabhakar derivative analyses with Caputo and Riemann derivative in [36]. The Hilfer-Prabhakar derivatives and their applications in time delay, time-fractional Poisson process its renewal structure discussed in [37]. Lyapunov Type inequality for hybrid fraction differential equation with Prabhakar derivative discussed in [38]. An accurate and robust numerical method is proportional to solve a class of nonlinear variable order FDE's in the Caputo-Prabhakar sense by using Bernstein polynomials in [39].

Fourier transform (FT) was defined by French mathematician Joseph Fourier. The term fraction power for the Fourier operators appears in the scientific field proposed by the winner in 1929. Fractional Fourier transforms (FrFT) is the generalization of the conventional Fourier transform. In 1980 Victor Nami initially introduced the FrFT gives the best outcomes for non-stationary signals than Fourier Transform due to its time-frequency characteristics. FrFT is used in several scientific fields such as optics, time-frequency distribution, image processing, satellite image compression signal image recovery and noise removal, image smoothing, encryption, and decryption. V. Namias as a way to solve FrFT plays a very important role in solving ordinary and partial equations. McBride and F. Kerr developed operational calculus for FrFT later refined his results. Recently of I Hyers- Ulam stability linear differential equations using Fourier transform proved by Murali Ramdos, Ponmana Selvan-Arumugam and Choonkil Park, Hamid Rezaei et al [40,41,42].

The analytical solutions of the fractional differential equation described by the fractional-order derivative operators is the main role in many stability problems. Motivated by the success of the application of the Mittag-Leffler functions in many areas of fractional calculus we present this paper. The main aim of this paper is to prove the Hyers-Ulam- Mittag-Leffler stability of the following fractional differential equations using the fractional Fourier transform.

$$\begin{cases} D_{\rho,\mu,\lambda}^{\gamma} u(t) + du(t) = q(t), & t^{\mu-1}u(t)|_{t=0} , & t \in (0, T] , d \in \mathbb{R}, \\ D_{\rho,\mu,\lambda}^{\gamma} u(t) + du(t) = G(t, u(t)), & t^{\mu-1}u(t)|_{t=0} , & t \in (0, T] , d \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} D_{\rho,\mu,\lambda}^{\gamma} u(t) + du(t) - q(t) = F(t), & t^{\mu-1}u(t)|_{t=0} , & t \in (0, T] , d \in \mathbb{R}, \\ D_{\rho,\mu,\lambda}^{\gamma} u(t) + du(t) - G(t, u(t)) = F(t), & t^{\mu-1}u(t)|_{t=0} , & t \in (0, T] , d \in \mathbb{R}, \end{cases}$$

The paper has been organized as follows. In Section 2, basic definitions, Theorems, and lemmas related to fractional derivatives, Mittag-Leffler function, Prabhakar derivatives, and Fractional Fourier Transform are given. In Section 3,4 Mittag-Leffler- Hyers-Ulam stability of linear and nonlinear fractional differential equations proved by using the Fractional Fourier transform. In Sections 5 and 6 examples and conclusions are given.

2. Preliminaries

Definition 2.1 The Riemann- Liouville Fractional differential operator

Suppose that $\theta > 0, t > a, \theta, a, t \in \mathbb{R}$ then

$$D^\theta f(t) = \begin{cases} \frac{1}{\Gamma(n-\theta)} \frac{d^n}{dt^n} \int_a^t \frac{f(T)}{(t-T)^{\theta+1-n}} dT, & n-1 < \theta < n, n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t), & \theta = n \in \mathbb{N} \end{cases}$$

is called the Riemann -Liouville derivative operator of order θ

Definition 2.2. The Mittag- Leffler Function

The one and two-parameter representations of the Mittag -Leffler function can be defined in terms of a power series as

$$E_\theta(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\theta k + 1)}, \theta > 0 \quad (2.1)$$

$$E_{\theta,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta + \theta k)}, \theta > 0, \beta > 0 \quad (2.2)$$

The exponential series defined by (2.1) gives the generalization of (2.2) it's introduced by R.P. Agarwal in 1953.

Definition 2.3 Prabhakar function

In 1970 the generalized Mittag-Leffler function with three- parameter is introduced by Prabhakar as

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)}$$

OR

$$E_{\alpha,\beta}^\gamma(z) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)}{k! \Gamma(\alpha k + \beta)} z^k$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}(\gamma) > 0$

Where $(\gamma)_k = \gamma(\gamma + 1)(\gamma + 2) \dots (\gamma + k - 1) = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)}, (\gamma)_0 = 1, \gamma \neq 0$.

is a Pochhammer symbol.

This function is also known as the Prabhakar function.

Definition 2.4 Prabhakar Integral

For $\in L'[a, b], 0 < t < b \leq \infty$, the Prabhakar integral operator including the generalized Mittag-Leffler function is defined as follows.

$$E_{\rho,\mu,w,0+}^\gamma f(t) = \int_0^t (t-u)^{u-1} E_{\rho,\mu}^\gamma(w(t-u)^\rho) f(u) du, t > 0$$

Where $\rho, \mu, w, \gamma \in \mathbb{C}, \operatorname{Re}(\rho), \operatorname{Re}(\mu) > 0$

Note 2.5 For $= 0$, the Prabhakar integral operator coincides with the Riemann-Liouville fractional integral of order μ , that is

$$E_{\rho,\mu,w,0+}^0 f = I_{0+}^\mu f$$

Definition 2.6 Prabhakar derivative

For $\in L'[a, b], 0 < t < b \leq \infty$, the Prabhakar derivative is defined by

$$D_{\rho,\mu,w,0+}^\gamma f(t) = \frac{d^m}{dt^m} E_{\rho,m-\mu,w,0+}^{-\gamma} f(t)$$

Where $\rho, \mu, w, \gamma \in \mathbb{C}, \operatorname{Re}(\rho), \operatorname{Re}(\mu) > 0$.

Note 2.7 It is obvious that the Prabhakar fractional derivative generates the Riemann-Liouville fractional derivative

$$D_{0+}^\mu f(t) = \frac{d^m}{dt^m} [I_{0+}^{m-\mu}] (t), m-1 < \mu < m..$$

Lemma 2.8 Let $\gamma, \rho, \mu, \lambda \in \mathbb{C}$ with $\operatorname{Re}(\mu) > 0$. Then for any $n \in \mathbb{N}$ differentiation of the generalized Mittag-Leffler function is given by

$$\left(\frac{d}{dx}\right)^n [x^{\mu-1} E_{\rho,\mu}^\gamma(\lambda x^\rho)] = x^{\mu-n-1} E_{\rho,\mu-n}^\gamma(\lambda x^\rho).$$

Lemma 2.9

The following series including the generalized Mittag-Leffler function with three parameters μ, ρ, γ is absolutely convergent series

$$S = \sum_{n=0}^{\infty} t^{\mu n} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^{\rho})$$

Where , $t \in \mathbb{C}, w, \rho \in \mathbb{R}, \mathbb{R}(\rho), \mathbb{R}(\mu) > 0$

Lemma 2.10

Suppose that

$C_{1-\mu}[a, T] = \{u(t) \in C[a, T]; t^{1-\mu}u(t) \in C[a, T]\}, 0 < \mu < 1$, and $u(t) \in C[a, T]$.

If there exists the limit

$$\lim_{t \rightarrow a^+} [(t-a)^{1-\mu}u(t)] = c, \quad c \in \mathbb{C},$$

Then the following relation holds:

$$(E_{\rho 1-\mu, \lambda, a^+}^{-\gamma} u)(a^+) = c \Gamma(\mu)$$

Also, if $(E_{\rho 1-\mu, \lambda, a^+}^{-\gamma} u)(a^+) = b, b \in \mathbb{C}$, and there exists the limit $\lim_{t \rightarrow a^+} [(t-a)^{1-\mu}u(t)]$, then

$$\lim_{t \rightarrow a^+} [(t-a)^{1-\mu}u(t)] = \frac{b}{\Gamma(\mu)}.$$

Definition 2.11 Fractional Fourier Transform

For a function, $u \in \Phi \in (\mathbb{R})$, the fractional Fourier transform of the order α ($0 < \alpha \leq 1$), $\widehat{u}_\alpha$ is defined as

$$\widehat{u}_\alpha = (\mathcal{F}_\alpha u)(\omega) = \int_{-\infty}^{\infty} u(t) e_\alpha(\omega, t) dt, \omega \in \mathbb{R}$$

$$\text{Where } e_\alpha(\omega, t) = \begin{cases} e^{-i|\omega|^{\frac{1}{\alpha}}t}, & \omega \leq 0 \\ e^{i|\omega|^{\frac{1}{\alpha}}t}, & \omega \geq 0 \end{cases}$$

If $\alpha = 1$, the kernel e_α is defined by considering with the kernel of the conventional Fourier transform

$$e_1(\omega, t) = \begin{cases} e^{-i|\omega|t}, & \omega \leq 0 \\ e^{i|\omega|t}, & \omega \geq 0 \end{cases} \equiv e^{i\omega t}, \omega \in \mathbb{R}, t \in \mathbb{R}$$

This means that the fractional Fourier transform of the order 1 conventional Fourier transform

Definition 2.12 One-to-one

Let $f, g \in \mathbb{R} \rightarrow \mathbb{F}$ be continuous and f^*, g^* are piece-wise continuous.

If $\mathbb{F}(f(x)) = \mathbb{F}(g(x))$ then $f(x) = g(x)$ for every x .

Definition 2.13 Convolution

The fractional Fourier transform of the convolution $f(x)$ and $g(x)$ is the product of the fractional Fourier transform of $f(x)$ and $g(x)$. Let $k, u \in \Phi(\mathbb{R})$ and $\alpha > 0$, then for any $\omega \in \mathbb{R}$, then

$$(\mathcal{F}_\alpha(k * u)(\omega)) = (\mathcal{F}_\alpha k)(\omega)(\mathcal{F}_\alpha u)(\omega)$$

In particular

$$(\mathcal{F}(k * u))(\omega) = (\mathcal{F}k)(\omega)(\mathcal{F}u)(\omega) (\omega \in \mathbb{R})$$

The convolution in the time domain is equivalent to multiplication in the frequency domain.

Result 2.14

Let $\beta \in \mathbb{C}$, $\mathbb{R}(\beta) > 0, t > 0$ then $\mathcal{F}_\alpha(t^\beta) = \frac{\Gamma(\beta+1)}{(-i\omega^{\frac{1}{\alpha}})^{\beta+1}}$

Theorem 2.15 Let $\gamma, \beta \in \mathbb{C}, \mathbb{R}(\gamma) > 0, \mathbb{R}(\beta) > 0, \lambda \in \mathbb{R}$ then hold

$$\mathcal{F}_\alpha(t^{\gamma m + \beta - 1} E_{\gamma, \beta}^{(m)}(\lambda t^\gamma)) = \frac{(-i\omega^{\frac{1}{\alpha}})^{\gamma - \beta} m!}{\left[(-i\omega^{\frac{1}{\alpha}})^\gamma - \lambda\right]^{m+1}}$$

Lemma 2.16

The Fractional Fourier Transform of Prabhakar integral is given by

$$F_\alpha[E_{\rho, \mu, \lambda, 0^+}^\gamma f(t)] = (-i\omega^{\frac{1}{\alpha}})^{-\mu} \left[1 - \lambda(-i\omega^{\frac{1}{\alpha}})^{-\gamma} F_\alpha(-i\omega^{\frac{1}{\alpha}})\right].$$

Proof:

$$\begin{aligned} F_\alpha[E_{\rho, \mu, \lambda, 0^+}^\gamma f(t)] &= F_\alpha \left[\int_0^t (t-u)^{\mu-1} E_{\rho, \mu}^\gamma[\lambda(t-u)^\rho] f(u) du \right] \\ &= F_\alpha[t^{\mu-1} E_{\rho, \mu}^\gamma(\lambda t^\rho)] F_\alpha[f(t)] \end{aligned}$$

$$\begin{aligned}
 &= F_{\alpha} \left[t^{\mu-1} \sum_{k=0}^{\infty} \frac{(\gamma_k)}{k!} \frac{(\lambda t^{\rho})^k}{\Gamma(\rho k + \mu)} \right] F_{\alpha}[f(t)] \\
 &= \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} \lambda^k \frac{F_{\alpha}[t^{\mu+\rho k-1}]}{\Gamma(\rho k + \mu)} F_{\alpha}[f(t)] \\
 &= \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} \frac{\lambda^k}{\Gamma(\rho k + \mu)} \frac{\Gamma(\rho k + \mu)}{\left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho k + \mu}} F_{\alpha}[f(t)] \\
 &= \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} \lambda^k \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho k} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} F_{\alpha}[f(t)] \\
 &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} \lambda^k \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho k} \right] F_{\alpha}[f(t)] \\
 &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[\frac{\gamma_0}{0!} \lambda^0 \left(-i\omega^{\frac{1}{\alpha}}\right)^0 + \frac{\gamma_1}{1!} \lambda^1 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho} + \frac{\gamma_2}{2!} \lambda^2 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-2\rho} + \frac{\gamma_3}{3!} \lambda^3 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-3\rho} + \dots \right] F_{\alpha}[f(t)] \\
 &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 + \frac{\gamma}{1!} \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho} + \frac{\gamma(\gamma+1)}{2!} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-2\rho} + \frac{\gamma(\gamma+1)(\gamma+2)}{3!} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-3\rho} + \dots \right] F_{\alpha}[f(t)] \\
 &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho} \right]^{-\gamma} F_{\alpha}[f(t)]
 \end{aligned}$$

ie $F_{\alpha}[E_{\rho,\mu,\lambda,0}^{\gamma} f(t)] = \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho} \right]^{-\gamma} F_{\alpha}[f(t)]$

Lemma 2.17

For $m-1 \leq \mu \leq m$ the Fractional Fourier Transform of Prabhakar derivative has the form

$$F_{\alpha}[D_{\rho,\mu,\lambda,0}^{\gamma} f(t)] = \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho} \right]^{\gamma} F_{\alpha}[f(t)]$$

Proof:

$$\begin{aligned}
 F_{\alpha}[D_{\rho,\mu,\lambda,0}^{\gamma} f(t)] &= F_{\alpha} \left[\frac{d^m}{dt^m} E_{\rho,m-\mu,\lambda,0}^{-\gamma} f(t) \right] \\
 &= F_{\alpha} \left[\frac{d^m}{dt^m} \int_0^t (t-u)^{m-\mu-1} E_{\rho,m-\mu,0}^{-\gamma} \lambda(t-\mu)^{\rho} f(u) du \right] \\
 &= F_{\alpha} \left[t^{m-\mu-1-m} E_{\rho,m-\mu-m,0}^{-\gamma} \lambda t^{\rho} \right] F_{\alpha}[f(t)] \\
 &= F_{\alpha} \left[t^{\mu-1} E_{\rho,-\mu}^{-\gamma} \lambda t^{\rho} \right] F_{\alpha}[f(t)] \\
 &= F_{\alpha} \left[t^{\mu-1} \sum_{k=0}^{\infty} \frac{(-\gamma)_k}{k!} \frac{(\lambda t^{\rho})^k}{\Gamma(\rho k + (-\mu))} \right] F_{\alpha}[f(t)] \\
 &= \sum_{k=0}^{\infty} \frac{(-\gamma)_k}{k!} \lambda^k \frac{F_{\alpha}(t^{-\mu+\rho k-1})}{\Gamma(\rho k + (-\mu))} F_{\alpha}[f(t)] \\
 &= \sum_{k=0}^{\infty} \frac{(-\gamma)_k}{k!} \frac{\lambda^k}{\Gamma(\rho k + (-\mu))} \frac{\Gamma(\rho k + (-\mu))}{\left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho k - \mu}} F_{\alpha}[f(t)] \\
 &= \sum_{k=0}^{\infty} \frac{(-\gamma)_k}{k!} \lambda^k \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho k} \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} F_{\alpha}[f(t)] \\
 &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[\sum_{k=0}^{\infty} \frac{(-\gamma)_k}{k!} \lambda^k \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho k} \right] F_{\alpha}[f(t)] \\
 &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[\frac{(-\gamma)_0}{0!} \lambda^0 \left(-i\omega^{\frac{1}{\alpha}}\right)^0 + \frac{(-\gamma)_1}{1!} \lambda^1 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho} + \frac{(-\gamma)_2}{2!} \lambda^2 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-2\rho} + \frac{(-\gamma)_3}{3!} \lambda^3 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-3\rho} \right. \\
 &\quad \left. + \dots \right] F_{\alpha}[f(t)]
 \end{aligned}$$

$$\begin{aligned}
 &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 + \frac{(-\gamma)}{1!} \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho} + \frac{(-\gamma)(-\gamma+1)}{2!} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-2\rho} + \frac{(-\gamma)(-\gamma+1)(-\gamma+2)}{3!} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-3\rho} \right. \\
 &\quad \left. + \dots \right] F_{\alpha}[f(t)] \\
 &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-(\gamma)} F_{\alpha}[f(t)] \\
 \text{ie } F_{\alpha}\left[E_{\rho,\mu,\lambda,0}^{\gamma} f(t)\right] &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} F_{\alpha}[f(t)]
 \end{aligned}$$

3. Mittag-Leffler- Hyers -Ulam Stability of fractional differential equations using Fractional Fourier Transform.

In this section, we are going to prove Mittag-Leffler- Hyer's-Ulam stability for linear and nonlinear fractional differential equations

$$\begin{cases} D_{\rho,\mu,\lambda}^{\gamma} u(t) + du(t) = q(t), & t^{\mu-1}u(t)|_{t=0}, \quad t \in (0, T], \quad d \in \mathbb{R}, \\ D_{\rho,\mu,\lambda}^{\gamma} u(t) + du(t) = G(t, u(t)), & t^{\mu-1}u(t)|_{t=0}, \quad t \in (0, T], \quad d \in \mathbb{R}, \end{cases}$$

with Prabhakar derivative using Fractional Fourier Transform.

3.1. Mittag-Leffler- Hyers -Ulam Stability of linear fractional differential equation

$$D_{\rho,\mu,\lambda}^{\gamma} u(t) + du(t) = q(t), \quad t^{\mu-1}u(t)|_{t=0}, \quad t \in (0, T], \quad d \in \mathbb{R}.$$

Definition 3.1 The fractional differential equation

$$\varphi(q(t), u(t), D_{\rho_1, \mu_1, \lambda}^{\gamma_1} u, \dots, D_{\rho_n, \mu_n, \lambda, 0}^{\gamma_n} u) = 0, \gamma_i \geq 0, i = 1, 2, \dots, n.$$

Has Hyers-Ulam stability if for given $\varepsilon > 0$ and a function μ such that

$$\left| \varphi(q(t), u(t), D_{\rho_1, \mu_1, \lambda}^{\gamma_1} u, \dots, D_{\rho_n, \mu_n, \lambda, 0}^{\gamma_n} u) \right| \leq \varepsilon,$$

There exists a solution u_{α} of the fractional differential equation such that

$$|u(t) - u_{\alpha}(t)| \leq K(\varepsilon), \lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0.$$

This statement is also true if we replace ε and $K(\varepsilon)$ by $F(t)$ and $C(t)$, where F and C are appropriate functions not depending on u and u_{α} explicitly.

Theorem 3.2

Let $G: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ $u(t) \in C_{1-\mu}[0, T]$. If the function $u(t)$ satisfies the inequality

$$|D_{\rho,\mu,\lambda}^{\gamma} u + du(t) - q(t)| \leq \varepsilon, 0 < \mu < 1, d \in \mathbb{R} \quad (3.1)$$

Then there exists a solution $u_{\alpha}(t): (0, T] \rightarrow \mathbb{C}$ of fractional differential equation

$$D_{\rho,\mu,\lambda}^{\gamma} u + d(t) = q(t) \quad (3.2)$$

Such that

$$|u(t) - u_{\alpha}(t)| \leq \frac{\varepsilon t^{\mu}}{|d|} \sum_{n=0}^{\infty} \frac{t^{\mu n} E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma}(|\lambda| t^{\rho})}{|d|^n}$$

Where $E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma}(|\lambda| t^{\rho})$ is the generalized Mittag- Leffler function.

Proof:

For all $t \in (0, T]$, we consider

$$y(t) = D_{\rho,\mu,\lambda}^{\gamma} u + du(t) - q(t)$$

Taking Fractional Fourier Transform on both side

$$\begin{aligned}
 F_{\alpha}[y(t)] &= F_{\alpha}[D_{\rho,\mu,\lambda}^{\gamma} u] + F_{\alpha}[du(t)] - F_{\alpha}[q(t)] \\
 &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} F_{\alpha}[u(t)] - \left(E_{\rho, 1-u, \lambda, 0}^{-\gamma} u\right)(0) + d F_{\alpha}[u(t)] - F_{\alpha}[q(t)]
 \end{aligned}$$

By using the lemma and consider $b = du_0 \Gamma(\mu)$ and $c = \frac{b}{d \Gamma(\mu)}$

$$\begin{aligned}
 &= \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} F_{\alpha}[u(t)] - du_0 \Gamma(\mu) + d F_{\alpha}[u(t)] - F_{\alpha}[q(t)] \\
 F_{\alpha}[y(t)] &= \left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d \right\} F_{\alpha}[u(t)] - du_0 \Gamma(\mu) - F_{\alpha}[q(t)]
 \end{aligned}$$

$$F_{\alpha}[y(t)] + du_0 \Gamma(\mu) + F_{\alpha}[q(t)] = \left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d \right\} F_{\alpha}[u(t)]$$

$$F_{\alpha}[u(t)] = \frac{F_{\alpha}[y(t)]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d \right\}} + \frac{du_0\Gamma(\mu) + F_{\alpha}[q(t)]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d \right\}}$$

At this point, by setting

$$u_a(t) = \frac{du_0\Gamma(\mu) \sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n t^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^{\rho})}{d} + \frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \int_0^t q(x)(t-x)^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda(t-x)^{\rho}) dx}{d} \quad (3.3)$$

We have

$$t^{1-\mu}u_a(t) = \frac{du_0\Gamma(\mu) \sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n t^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^{\rho})}{d} + \frac{t^{1-\mu} \sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \int_0^t q(x)(t-x)^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda(t-x)^{\rho}) dx}{d}$$

This implies that $u_a(t)$ satisfies the initial value condition when

$$t^{1-\mu}u_a(t)|_{t=0}$$

We now apply the Fractional Fourier Transform on both sides of relation and use the convolution property of the fractional Fourier transform we obtain

$$F_{\alpha}[u_a(t)] = F_{\alpha} \left[\frac{du_0\Gamma(\mu) \sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n t^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^{\rho})}{d} \right] + F_{\alpha} \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \int_0^t q(x)(t-x)^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda(t-x)^{\rho}) dx}{d} \right]$$

$$F_{\alpha}[u_a(t)] = \frac{du_0\Gamma(\mu) \sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n}{d} F_{\alpha}[t^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^{\rho})] + \frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n}{d} F_{\alpha} \left[\int_0^t q(x)(t-x)^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda(t-x)^{\rho}) dx \right]$$

$$F_{\alpha}[u_a(t)] = \frac{du_0\Gamma(\mu) \sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n}{d} F_{\alpha}[t^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^{\rho})] + \frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n}{d} F_{\alpha}[t^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^{\rho})] F_{\alpha}[q(t)]$$

$$F_{\alpha}[u_a(t)] = du_0\Gamma(\mu) + F_{\alpha}[q(t)] \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n F_{\alpha}[t^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^{\rho})]}{d} \right]$$

$$F_{\alpha}[u_a(t)] = du_0\Gamma(\mu) + F_{\alpha}[q(t)] \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho(\gamma n + \gamma) - (\mu n + \mu)}}{d \left[\left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho} - \lambda \right]^{\gamma n + \gamma}} \right]$$

$$= du_0\Gamma(\mu) + F_{\alpha}[q(t)] \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho(\gamma n + \gamma) - (\mu n + \mu)}}{d \left[\left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} \right]^{\gamma n + \gamma}} \right]$$

$$\begin{aligned}
 &= du_0\Gamma(\mu) + F_\alpha[q(t)] \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho\gamma(n+1)} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu(n+1)}}{d \left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho\gamma(n+1)} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma(n+1)}} \right] \\
 &= [du_0\Gamma(\mu) + F_\alpha[q(t)]] \frac{1}{d} \left[\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu(n+1)} \left(1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right)^{\gamma(n+1)} \right] \\
 &= [du_0\Gamma(\mu) + F_\alpha[q(t)]] \frac{1}{d} \left[\left(\frac{-1}{d}\right)^0 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \right. \\
 &\quad \left. + \left(\frac{-1}{d}\right)^1 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-2\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-2\gamma} + \left(\frac{-1}{d}\right)^2 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-3\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-3\gamma} \right. \\
 &\quad \left. + \dots \right] \\
 &= [du_0\Gamma(\mu) + F_\alpha[q(t)]] \frac{1}{d} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \left[1 + \left(\frac{-1}{d}\right)^1 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \right. \\
 &\quad \left. + \left(\frac{-1}{d}\right)^2 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-2\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-2\gamma} + \dots \right] \\
 &= [du_0\Gamma(\mu) + F_\alpha[q(t)]] \frac{1}{d} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \left[\frac{1}{d} + \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \right]^{-1} \\
 &= [du_0\Gamma(\mu) + F_\alpha[q(t)]] \frac{1}{d} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \left[\frac{1}{d} + \frac{1}{\left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma}} \right]^{-1} \\
 &= [du_0\Gamma(\mu) + F_\alpha[q(t)]] \frac{1}{d} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \left[\frac{\left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d}{d \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma}} \right]^{-1} \\
 &= [du_0\Gamma(\mu) + F_\alpha[q(t)]] \frac{1}{d} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \left[\frac{d \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma}}{\left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d} \right] \\
 &F_\alpha[u_a(t)] = \frac{du_0\Gamma(\mu) + F_\alpha[q(t)]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d \right\}}
 \end{aligned}$$

(3.4)

And taking (3.4) into account, we deduce that $u_a(t)$ is a solution of (3.2). There, by subtracting relations(3.3) and (3.4) from each other we get

$$\begin{aligned}
 F_\alpha[u(t)] - F_\alpha[u_a(t)] &= \frac{F_\alpha[y(t)] + du_0\Gamma(\mu) + F_\alpha[q(t)] - [du_0\Gamma(\mu) + F_\alpha[q(t)]]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d \right\}} \\
 F_\alpha[u(t)] - F_\alpha[u_a(t)] &= \frac{F_\alpha[y(t)]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d \right\}}
 \end{aligned}$$

(3.5)

By lemma(2.16) right hand side of the above equation we have

$$F_\alpha \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n t^{\mu n} E_{\rho, \mu n + \mu}^{\gamma n + \gamma} (\lambda t^\rho)}{d} * y(t) \right] = F_\alpha \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n t^{\mu n} E_{\rho, \mu n + \mu}^{\gamma n + \gamma} (\lambda t^\rho)}{d} \right] F_\alpha[y(t)]$$

$$= \frac{F_\alpha[y(t)]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}} \right)^\mu \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}} \right)^{-\rho} \right]^\gamma + d \right\}}$$

Hence the relation (3.5) can be written as

$$F_\alpha[u(t)] - F_\alpha[u_\alpha(t)] = F_\alpha \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d} \right)^n t^{\mu n} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^\rho)}{d} * y(t) \right]$$

Finally, from (3.1) it follows that

$$\begin{aligned} |u(t) - u_\alpha(t)| &= \left| \frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d} \right)^n t^{\mu n} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^\rho)}{d} * y(t) \right| \\ &= \left| \frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d} \right)^n}{d} \int_0^t y(x)(t-x)^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda(t-x)^\rho) dx \right| \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left| \frac{(\gamma n + \gamma)_k \lambda^k \int_0^t y(x)(t-x)^{\mu n + \mu - 1} dx}{(-d)^{n+1} k! \Gamma(\rho k + \mu n + \mu)} \right| \\ &\leq \varepsilon \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma n + \gamma)_k |\lambda|^k}{|d|^{n+1} k! \Gamma(\rho k + \mu n + \mu)} \int_0^t |(t-x)^{\rho k + \mu n + \mu - 1}| dx \\ &= \varepsilon \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma n + \gamma)_k |\lambda|^k}{|d|^{n+1} k! \Gamma(\rho k + \mu n + \mu)} \int_0^t (t-x)^{\rho k + \mu n + \mu - 1} dx \\ &= \varepsilon \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma n + \gamma)_k |\lambda|^k}{|d|^{n+1} k! \Gamma(\rho k + \mu n + \mu)} \left[\frac{(t-x)^{\rho k + \mu n + \mu}}{\rho k + \mu n + \mu (-1)} \right]_{x=0}^{x=t} \\ &= \varepsilon \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma n + \gamma)_k |\lambda|^k}{|d|^{n+1} k! \Gamma(\rho k + \mu n + \mu)} \left[\frac{t^{\rho k + \mu n + \mu}}{\rho k + \mu n + \mu} \right] \\ &= \frac{\varepsilon t^\mu}{|d|} \sum_{n=0}^{\infty} \frac{t^{\mu n} E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma}(|\lambda| t^\rho)}{|d|^n} \end{aligned}$$

Where $E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma}(|\lambda| t^\rho)$ is the generalized Mittag-Leffler function.

By the definition of the Hyers-Ulam stability theorem, the fractional differential equation 3.1 has Mittag-Leffler- Hyers -Ulam Stability.

3.2. Mittag-Leffler- Hyers -Ulam Stability of Nonlinear fractional differential equation

$$D_{\rho, \mu, \lambda}^\gamma u(t) + du(t) = G(t, u(t)), \quad t^{\mu-1}u(t)|_{t=0}, \quad t \in (0, T], \quad d \in \mathbb{R}$$

Definition 3.3

For $G: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, the fractional differential equation

$$\varphi \left(G(t, u(t)), u(t), D_{\rho_1, \mu_1, \lambda}^{\gamma_1} u, \dots, D_{\rho_n, \mu_n, \lambda, 0^+}^{\gamma_n} u \right) = 0, \quad \gamma_i \geq 0, i = 1, 2, \dots, n.$$

Has Hyers-Ulam stability if for given $\varepsilon > 0$ and a function μ such that

$$\left| \varphi \left(G(t, u(t)), u(t), D_{\rho_1, \mu_1, \lambda}^{\gamma_1} u, \dots, D_{\rho_n, \mu_n, \lambda, 0^+}^{\gamma_n} u \right) \right| \leq \varepsilon,$$

There exist a solution u_α of the fractional differential equation such that

$$|u(t) - u_\alpha(t)| \leq K(\varepsilon), \lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0.$$

This statement is also true if we replace ε and $K(\varepsilon)$ by $F(t)$ and $C(t)$, where F and C are appropriate functions not depending on u and u_α explicitly.

Theorem 3.4

Let $G: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ $u(t) \in C_{1-\mu}[0, T]$. If the function $u(t)$ satisfies the inequality

$$|D_{\rho, \mu, w}^\gamma u + du(t) - G(t, u(t))| \leq \varepsilon, 0 < \mu < 1,$$

Then there exists a solution $u_\alpha(t): (0, T] \rightarrow \mathbb{C}$ of fractional differential equation

$$D_{\rho, \mu, w}^\gamma u + d(t) = G(t, u(t))$$

Such that

$$|u(t) - u_\alpha(t)| \leq \frac{\varepsilon t^\mu}{|d|} \sum_{n=0}^{\infty} \frac{t^{\mu n E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma} (|\lambda| t^\rho)}}{|d|^n}$$

Where $E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma} (|\lambda| t^\rho)$ is the generalized Mittag- Leffler function.

Proof:

The proof is similar to that of theorem 3.2. the only difference is to replace the function $q(t)$ by the function $G(t, u(t))$.

4. Mittag-Leffler- Hyers -Ulam Stability of fractional differential equations using Fractional Differential equations.

In this section, we are going to prove Mittag-Leffler- Hyer's-Ulam stability for linear and nonlinear fractional differential equations

$$\begin{cases} D_{\rho, \mu, \lambda}^\gamma u(t) + du(t) - q(t) = F(t), & t^{\mu-1}u(t)|_{t=0}, & t \in (0, T], d \in \mathbb{R}, \\ D_{\rho, \mu, \lambda}^\gamma u(t) + du(t) - G(t, u(t)) = F(t), & t^{\mu-1}u(t)|_{t=0}, & t \in (0, T], d \in \mathbb{R}, \end{cases}$$

With Prabhakar derivative using Fractional Fourier Transform.

4.1. Mittag-Leffler- Hyers -Ulam Stability of linear fractional differential equation

$$D_{\rho, \mu, \lambda}^\gamma u(t) + du(t) - q(t) = F(t), \quad t^{\mu-1}u(t)|_{t=0}, \quad t \in (0, T], d \in \mathbb{R}$$

Theorem 4.1

For all $t \in C_{1-\mu}[0, T]$. and some $F(t) > 0$, if a function $u(t)$ satisfies the inequality

$$|D_{\rho, \mu, \lambda}^\gamma u + du(t) - q(t)| \leq F(t), 0 < \mu < 1, d \in \mathbb{R} \quad (4.1)$$

The there exist a solution $u_\alpha(t): (0, T] \rightarrow \mathbb{C}$ of the fractional differential Equation

$$D_{\rho, \mu, \lambda}^\gamma u + du(t) - q(t) = F(t) \quad (4.2)$$

Such that

$$|u(t) - u_\alpha(t)| \leq \frac{F(t)}{|d|} \sum_{n=0}^{\infty} \frac{t^{\mu n E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma} (|\lambda| t^\rho)}}{|d|^n}$$

Proof:

For all $t \in (0, T]$, we consider

$$y(t) = D_{\rho, \mu, \lambda}^\gamma u + du(t) - q(t) - F(t)$$

Taking Fractional Fourier Transform on both side

$$\begin{aligned} F_\alpha[y(t)] &= F_\alpha[D_{\rho, \mu, \lambda}^\gamma u] + F_\alpha[du(t)] - F_\alpha[q(t)] - F_\alpha[F(t)] \\ &= \left(-i\omega^{\frac{1}{\alpha}}\right)^\mu \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^\gamma F_\alpha[u(t)] - \left(E_{\rho, 1-u, \lambda, 0}^{-\gamma} u\right)(0) + dF_\alpha[u(t)] - F_\alpha[q(t)] - F_\alpha[F(t)] \end{aligned}$$

By using the lemma and consider $b = du_0 \Gamma(\mu)$ and $c = \frac{b}{d \Gamma(\mu)} t$

$$\begin{aligned} &= \left(-i\omega^{\frac{1}{\alpha}}\right)^\mu \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^\gamma F_\alpha[u(t)] - du_0 \Gamma(\mu) + dF_\alpha[u(t)] - F_\alpha[q(t)] - F_\alpha[F(t)] \\ F_\alpha[y(t)] &= \left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^\mu \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^\gamma + d \right\} F_\alpha[u(t)] - du_0 \Gamma(\mu) - F_\alpha[q(t)] - F_\alpha[F(t)] \end{aligned}$$

$$F_\alpha[y(t)] + du_0 \Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)] = \left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^\mu \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^\gamma + d \right\} F_\alpha[u(t)]$$

$$F_\alpha[u(t)] = \frac{F_\alpha[y(t)]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^\mu \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^\gamma + d \right\}} + \frac{du_0 \Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^\mu \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^\gamma + d \right\}}$$

At this point ,by setting

$$\begin{aligned} u_\alpha(t) &= \frac{du_0 \Gamma(\mu) \sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n t^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma} (\lambda t^\rho)}{d} \\ &+ \frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \int_0^t q(x) (t-x)^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma} (\lambda (t-x)^\rho) dx}{d} \\ &+ \frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \int_0^t F(x) (t-x)^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma} (\lambda (t-x)^\rho) dx}{d} \end{aligned} \quad (4.3)$$

We have

$$t^{1-\mu}u_a(t) = \frac{du_0\Gamma(\mu)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n t^{\mu n}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda t^\rho)}{d} \\ + \frac{t^{1-\mu}\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n \int_0^t q(x)(t-x)^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda(t-x)^\rho)dx}{d} \\ + \frac{t^{1-\mu}\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n \int_0^t F(x)(t-x)^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda(t-x)^\rho)dx}{d}$$

Which implies that $u_a(t)$ satisfies the initial value condition when

$$t^{1-\mu}u_a(t)|_{t=0}$$

We now apply the Fractional Fourier Transform on both sides of relation and use the convolution property of the fractional Fourier transform we obtain

$$F_\alpha[u_a(t)] = F_\alpha\left[\frac{du_0\Gamma(\mu)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n t^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda t^\rho)}{d}\right] \\ + F_\alpha\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n \int_0^t q(x)(t-x)^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda(t-x)^\rho)dx}{d}\right] \\ + F_\alpha\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n \int_0^t F(x)(t-x)^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda(t-x)^\rho)dx}{d}\right] \\ F_\alpha[u_a(t)] = \frac{du_0\Gamma(\mu)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d}F_\alpha[t^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda t^\rho)] \\ + \frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d}F_\alpha\left[\int_0^t q(x)(t-x)^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda(t-x)^\rho)dx\right] \\ + \frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d}F_\alpha\left[\int_0^t F(x)(t-x)^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda(t-x)^\rho)dx\right] \\ F_\alpha[u_a(t)] = \frac{du_0\Gamma(\mu)\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d}F_\alpha[t^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda t^\rho)] \\ + \frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d}F_\alpha[t^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda t^\rho)]F_\alpha[q(t)] + \frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n}{d}F_\alpha[t^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda t^\rho)]F_\alpha[F(t)] \\ F_\alpha[u_a(t)] = \{du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]\}\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n F_\alpha[t^{\mu n+\mu-1}E_{\rho,\mu n+\mu}^{\gamma n+\gamma}(\lambda t^\rho)]}{d}\right] \\ F_\alpha[u_a(t)] = \{du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]\}\left[\frac{\sum_{n=0}^{\infty}\left(\frac{-1}{d}\right)^n (-i\omega^{\frac{1}{\alpha}})^{\rho(\gamma n+\gamma)-(\mu n+\mu)}}{d \left[(-i\omega^{\frac{1}{\alpha}})^\rho - \lambda\right]^{\gamma n+\gamma}}\right]$$

$$\begin{aligned}
 &= \{du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]\} \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho(\gamma n + \gamma) - (\mu n + \mu)}}{d \left[\left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma n + \gamma} \right]} \right] \\
 &= \{du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]\} \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho\gamma(n+1)} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu(n+1)}}{d \left(-i\omega^{\frac{1}{\alpha}}\right)^{\rho\gamma(n+1)} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma(n+1)}} \right] \\
 &= \{du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]\} \frac{1}{d} \left[\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu(n+1)} \left(1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right)^{\gamma(n+1)} \right] \\
 &= \{du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]\} \frac{1}{d} \left[\left(\frac{-1}{d}\right)^0 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \right. \\
 &\quad \left. + \left(\frac{-1}{d}\right)^1 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-2\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-2\gamma} + \left(\frac{-1}{d}\right)^2 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-3\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-3\gamma} \right. \\
 &\quad \left. + \dots \right] \\
 &= [du_0\Gamma(\mu) + F_\alpha[q(t)]] \frac{1}{d} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \left[1 + \left(\frac{-1}{d}\right)^1 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \right. \\
 &\quad \left. + \left(\frac{-1}{d}\right)^2 \left(-i\omega^{\frac{1}{\alpha}}\right)^{-2\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-2\gamma} + \dots \right] \\
 &= \{du_0\Gamma(\mu) + F_\alpha[q(t)] \\
 &\quad + F_\alpha[F(t)]\} \frac{1}{d} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \left[\frac{1}{d} + \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \right]^{-1} \\
 &= \{du_0\Gamma(\mu) + F_\alpha[q(t)] \\
 &\quad + F_\alpha[F(t)]\} \frac{1}{d} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \left[\frac{1}{d} + \frac{1}{\left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma}} \right]^{-1} \\
 &= \{du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]\} \frac{1}{d} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \left[\frac{\left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d}{d \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma}} \right]^{-1} \\
 &= \{du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]\} \frac{1}{d} \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{-\gamma} \left[\frac{d \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma}}{\left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d} \right] \\
 F_\alpha[u_a(t)] &= \frac{du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d \right\}} \\
 (4.4)
 \end{aligned}$$

And taking (4.4) into account, we deduce that $u_a(t)$ is a solution of (4.2). There, by subtracting relations(4.3) and (4.4) from each other we get

$$F_\alpha[u(t)] - F_\alpha[u_a(t)] = \frac{F_\alpha[y(t)] + du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)] - [du_0\Gamma(\mu) + F_\alpha[q(t)] + F_\alpha[F(t)]]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^{\mu} \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^{\gamma} + d \right\}}$$

$$F_\alpha[u(t)] - F_\alpha[u_\alpha(t)] = \frac{F_\alpha[y(t)]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^\mu \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^\gamma + d \right\}}$$

By lemma right hand side of the above equation we have

$$\begin{aligned} F_\alpha \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n t^{\mu n} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^\rho)}{d} * y(t) \right] &= F_\alpha \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n t^{\mu n} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^\rho)}{d} \right] F_\alpha[y(t)] \\ &= \frac{F_\alpha[y(t)]}{\left\{ \left(-i\omega^{\frac{1}{\alpha}}\right)^\mu \left[1 - \lambda \left(-i\omega^{\frac{1}{\alpha}}\right)^{-\rho}\right]^\gamma + d \right\}} \end{aligned}$$

Hence the relation (3.5) can be written as

$$F_\alpha[u(t)] - F_\alpha[u_\alpha(t)] = F_\alpha \left[\frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n t^{\mu n} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^\rho)}{d} * y(t) \right]$$

Finally, from (3.1) it follows that

$$\begin{aligned} |u(t) - u_\alpha(t)| &= \left| \frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n t^{\mu n} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda t^\rho)}{d} * y(t) \right| \\ &= \left| \frac{\sum_{n=0}^{\infty} \left(\frac{-1}{d}\right)^n}{d} \int_0^t y(x)(t-x)^{\mu n + \mu - 1} E_{\rho, \mu n + \mu}^{\gamma n + \gamma}(\lambda(t-x)^\rho) dx \right| \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left| \frac{(\gamma n + \gamma)_k \lambda^k \int_0^t y(x)(t-x)^{\mu n + \mu - 1} dx}{(-d)^{n+1} k! \Gamma(\rho k + \mu n + \mu)} \right| \\ &\leq F(t) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma n + \gamma)_k |\lambda|^k}{|d|^{n+1} k! \Gamma(\rho k + \mu n + \mu)} \int_0^t |(t-x)^{\rho k + \mu n + \mu - 1}| dx \\ &= F(t) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma n + \gamma)_k |\lambda|^k}{|d|^{n+1} k! \Gamma(\rho k + \mu n + \mu)} \int_0^t (t-x)^{\rho k + \mu n + \mu - 1} dx \\ &= F(t) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma n + \gamma)_k |\lambda|^k}{|d|^{n+1} k! \Gamma(\rho k + \mu n + \mu)} \left[\frac{(t-x)^{\rho k + \mu n + \mu}}{\rho k + \mu n + \mu (-1)} \right]_{x=0}^{x=t} \\ &= F(t) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma n + \gamma)_k |\lambda|^k}{|d|^{n+1} k! \Gamma(\rho k + \mu n + \mu)} \left[\frac{t^{\rho k + \mu n + \mu}}{\rho k + \mu n + \mu} \right] \\ &= \frac{F(t) t^\mu}{|d|} \sum_{n=0}^{\infty} \frac{t^{\mu n} E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma}(|\lambda| t^\rho)}{|d|^n} \end{aligned}$$

Where $E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma}(|\lambda| t^\rho)$ is the generalized Mittag-Leffler function.

By the definition of Hyers-Ulam stability theorem, the fractional differential equation 3.1 has Mittag-Leffler- Hyers -Ulam Stability.

4.2. Mittag-Leffler- Hyers -Ulam Stability of Non-linear fractional differential equation

$$D_{\rho, \mu, \lambda}^{\gamma} u(t) + du(t) - G(t, u(t)) = F(t), \quad t^{\mu-1} u(t)|_{t=0}, \quad t \in (0, T], d \in \mathbb{R}$$

Theorem 4.2

Let $G: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ $u(t) \in C_{1-\mu}[0, T]$. If the function $u(t)$ satisfies the inequality

$$|D_{\rho, \mu, w}^{\gamma} u + du(t) - G(t, u(t))| \leq F(t), 0 < \mu < 1,$$

Then there exists a solution $u_\alpha(t): (0, T] \rightarrow \mathbb{C}$ of fractional differential equation

$$D_{\rho, \mu, w}^{\gamma} u + d(t) - G(t, u(t)) = F(t)$$

Such that

$$|u(t) - u_\alpha(t)| \leq \frac{F(t) t^\mu}{|d|} \sum_{n=0}^{\infty} \frac{t^{\mu n} E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma}(|\lambda| t^\rho)}{|d|^n}$$

Where $E_{\rho, \mu n + \mu + 1}^{\gamma n + \gamma}(|\lambda| t^\rho)$ is the generalized Mittag- Leffler function.

Proof:

The proof is similar to that of theorem 4.1. the only difference is to replace the function $q(t)$ with the function $G(t, u(t))$.

5.Example

1.Let us consider the fractional differential Equation

$$D_{1, \frac{1}{2}, 1}^1 u(t) + 6u(t) = \frac{5\sqrt{e^t}}{2} + 6t^{\frac{1}{2}} + \frac{1}{16}$$

Where $\mu = 1, \rho = 1, \delta = 1$ and $q(t) = \frac{5\sqrt{e^t}}{2} + 6t^{\frac{1}{2}} + \frac{1}{16}$.

For $\varepsilon = \frac{1}{8}$, the function $u(t)$ satisfies

$$\left| D_{1, \frac{1}{2}, 1}^1 u(t) + 6u(t) - \frac{5\sqrt{e^t}}{2} - 6t^{\frac{1}{2}} - \frac{1}{16} \right| < \frac{1}{8}$$

And the initial condition is $t^{\frac{1}{2}}u(t)\Big|_{t=0}$. By using (), an exact solution of () can be written as $u_\alpha(t) =$

$$\frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{-1}{7}\right)^n \int_0^t \left(\frac{5\sqrt{e^x}}{2} + 6x^{\frac{1}{2}} + \frac{1}{16}\right) (t-x)^{\frac{1}{2}n - \frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}n + \frac{3}{2}}^{n+1}(t-x) dx,$$

And the error of the approximate solution $u_1(t)$ is given by

$$\left| u_1(t) - u_\alpha(t) \right| < \frac{\varepsilon t^{\frac{1}{2}}}{|d|} \sum_{n=0}^{\infty} \frac{t^{\frac{1}{2}n} E_{\frac{1}{2}, \frac{1}{2}n + \frac{3}{2}}^{n+1}(t)}{|6|^n}$$

So that

$$\frac{t^{\frac{1}{2}}}{48} \sum_{n=0}^{\infty} \frac{t^{\frac{1}{2}n} E_{\frac{1}{2}, \frac{1}{2}n + \frac{3}{2}}^{n+1}(t)}{|6|^n}$$

Is the control function of $u_1(t)$

6.CONCLUSION

In this paper this research analysis Hyers-Ulam-Mittag-Leffler's stability of linear and non-linear fractional differential equations with Prabhakar derivative using Fractional Fourier transform. By proven results, we show that fractional differential equations are stable in the sense of "Mittag-Leffler-Hyers-Ulam". Some examples are given to illustrate the main theoretical results.

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