



ON THE ULAM-GAVRUTA-RASSIAS STABILITY OF THE ORTHOGONALLY EULER-LAGRANGE TYPE FUNCTIONAL EQUATION

¹M.Settu, ²G. Balasubramanian

^{1,2}Department of Mathematics,

Government Arts College (For Men), Krishnagiri -635 00. India.

settum87@gmail.com

gbs_geetha@yahoo.com

Corresponding Author: M.Settu

DOI: <https://doi.org/10.64971/j.cph.eijtem.v13.i4.3.2026>

Abstract:

In this paper, we prove the Hyers-Ulam stability of the orthogonally n-dimensional quadratic functional equation of the form

$$g\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n g\left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) = (n-3) \sum_{1 \leq i < j \leq n} g(x_i + x_j) + (-n^2 + 5n - 2) \sum_{i=1}^n g(x_i)$$

where n is a positive integer with $n \geq 3$.

Keywords: Hyers-Ulam stability, Ulam-Gavruta-Rassias stability, Orthogonally Euler-Lagrange functional equation, Quadratic mapping.

AMS Subject Classification: 39B55, 39B52, 39B82, 46H25.

1.Introduction

In 1982, J. M. Rassias[13] followed the innovative approach of the Th. M. Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \|y\|^p$ for $p, q \in \mathbb{R}$ with $p + q = 1$. A generalization of all the above results was obtained by P. Gavruta[5] in 1994 by replacing the unbounded Cauchy difference by a general control function $\phi(x, y)$ in the spirit of Rassias approach.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi et al., by considering the summation of both the sum and the product of two p-norms in the spirit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results

concerning this problem (see[1,2,3,4,6,10,12,15] . Some of the functional papers are used to develop this paper which are [5,7,8,9,14].

Definition 1.1 A vector space X is called an orthogonally vector space if there is a relation $x \perp y$ on X such that

- (i) $x \perp 0, 0 \perp x$ for all $x \in X$;
- (ii) If $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;
- (iii) $x \perp y$, then $ax \perp by$ for all $a, b \in \mathbb{R}$;
- (iv) If P is an two-dimensional subspace of X ; then
 - (a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;
 - (b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.

Any vector space can be made into an orthogonally vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector x, y define $x \perp y$ iff x, y are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. The pair $(X, \perp)$ is called an orthogonality space. It becomes orthogonality normed space when the orthogonality space equipped with a norm.

In this paper, we investigate n -dimensional Euler-Lagrange functional equation of the form

$$g\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n g\left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) = (n-3) \sum_{1 \leq i < j \leq n} g(x_i + x_j) + (-n^2 + 5n - 2) \sum_{i=1}^n g(x_i) \quad (1.1)$$

and study the Hyers-Ulam stability in the concept of orthogonality and Ulam-Gavruta-Rassias stability.

A mapping $g: X \rightarrow Y$ is called orthogonal quadratic if it satisfies the quadratic functional equation (1.1) for all $x_1, x_2, \dots, x_n \in X$ with $x_1 \perp x_2 \perp \dots \perp x_n$ where X be an orthogonality space and Y be a real banach space.

2. Stability of the Orthogonally Euler-Lagrange Type Functional Equation (1.1) involving Sum of Powers of Norms

In this section, let $(X, \perp)$ denotes an orthogonality normed space with norm $\|\cdot\|_X$ and $(Y, \|\cdot\|_Y)$ is a Banach space.

Theorem 2.1 Let λ and $s (s < 2)$ be non-negative real numbers. Let $g: X \rightarrow Y$ be a mapping fulfilling

$$\left\| g\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n g\left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) - (n-3) \sum_{1 \leq i < j \leq n} g(x_i + x_j) - (-n^2 + 5n - 2) \sum_{i=1}^n g(x_i) \right\|_Y$$

$$\leq \lambda \left\{ \|x_1\|_X^s + \|x_2\|_X^s + \dots + \|x_n\|_X^s \right\} \quad (2.1)$$

for all $x_1, x_2, \dots, x_n \in X$ with $x_1 \perp x_2 \perp \dots \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A: X \rightarrow Y$ such that

$$\|g(x) - A(x)\|_Y \leq \frac{\lambda \|x\|_X^s}{2(n-5)(2^2 - 2^s)} \quad (2.2)$$

for all $x \in X$.

Proof. Setting $(x_1, x_2, \dots, x_n)$ by $(0, 0, \dots, 0)$ in (2.1), we have $f(0) = 0$. Letting Replacing $(x_1, x_2, \dots, x_n)$ by $(x, -x, x, -x, x, 0, \dots, 0)$ in (2.1), we obtain

$$\|2(n-5)g(2x) - 8(n-5)g(x)\|_Y \leq \lambda \left\{ \|x\|_X^p \right\} \quad (2.3)$$

for all $x \in X$. Since $x \perp 0$, we get

$$\left\| \frac{g(2x)}{4} - g(x) \right\|_Y \leq \frac{1}{8(n-5)} \lambda \left\{ \|x\|_X^p \right\} \quad (2.4)$$

for all $x \in X$. Now replacing x by $2x$ and divided by 4 in (2.4) and summing the resulting inequality with (2.4), we obtain

$$\left\| \frac{g(2^2 x)}{2^4} - g(x) \right\|_Y \leq \frac{\lambda}{8(n-5)} \left\{ 1 + \frac{2^s}{2^2} \right\} \|x\|_X^s \quad (2.5)$$

for all $x \in X$. Using the induction on n , we receive

$$\left\| \frac{g(2^n x)}{2^{2n}} - g(x) \right\|_Y \leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{n-1} \frac{2^{sk}}{2^{2k}} \|x\|_X^s \leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{\infty} \frac{2^{sk}}{2^{2k}} \|x\|_X^s \quad (2.6)$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \frac{g(2^n x)}{2^{2n}} \right\}$ replace x by $2^m x$

and divided by 2^{2m} in (2.6), for any $n, m > 0$ we get

$$\begin{aligned}
 \left\| \frac{g(2^{n+m}x)}{2^{2(n+m)}} - \frac{g(2^m x)}{2^{2m}} \right\|_Y &= \frac{1}{2^{2m}} \left\| \frac{g(2^{n+m}x)}{2^{2n}} - g(2^m x) \right\|_Y \\
 &\leq \frac{\lambda}{8(n-5)2^{2m}} \sum_{k=0}^{n-1} \frac{2^{sk}}{2^{2k}} \|2^m x\|_X^s \\
 &\leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{\infty} \frac{2^{s(k+m)}}{2^{2(k+m)}} \|x\|_X^s \\
 &\leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{\infty} \frac{1}{2^{(2-s)(k+m)}} \|x\|_X^s \quad (2.7)
 \end{aligned}$$

As $s < 2$, the R.H.S. of (2.7) tends to 0 as $m \rightarrow \infty$ for all $x \in X$. Thus $\left\{ \frac{g(2^n x)}{2^{2n}} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X \rightarrow Y$ such that

$$A(x) = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{2^{2n}}, \quad \forall x \in X.$$

Taking $n \rightarrow \infty$ in (2.6), we have at the formula (2.2) for all $x \in X$. To prove A satisfies (1.1), replace $(x_1, x_2, \dots, x_n)$ by $(2^n x_1, 2^n x_2, \dots, 2^n x_n)$ in (2.1) and divided by 2^{2n} then it follows that

$$\begin{aligned}
 \frac{1}{2^{2n}} \left\| g\left(\sum_{i=1}^n 2^n x_i\right) + \sum_{j=1}^n g\left(2^n \left(-x_j + \sum_{i=1, i \neq j}^n x_i\right)\right) - (n-3) \sum_{1 \leq i < j \leq n} g(2^n (x_i + x_j)) - (-n^2 + 5n - 2) \sum_{i=1}^n g(2^n x_i) \right\|_Y \\
 \leq \frac{\lambda}{2^{2n}} \left\{ \|2^n x_1\|_X^s + \|2^n x_2\|_X^s + \dots + \|2^n x_n\|_X^s \right\}.
 \end{aligned}$$

Again taking limit as $n \rightarrow \infty$ in the above inequality, we have

$$\left\| A\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n A\left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) - (n-3) \sum_{1 \leq i < j \leq n} A(x_i + x_j) - (-n^2 + 5n - 2) \sum_{i=1}^n A(x_i) \right\|_Y = 0$$

which gives

$$g\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n g\left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) = (n-3) \sum_{1 \leq i < j \leq n} g(x_i + x_j) + (-n^2 + 5n - 2) \sum_{i=1}^n g(x_i)$$

for all $x_1, x_2, \dots, x_n \in X$ with $x_1 \perp x_2 \perp \dots \perp x_n$. Therefore $A: X \rightarrow Y$ is an orthogonally quadratic mapping which satisfies (1.1).

To prove the uniqueness of A. Let B be another quadratic function satisfying (1.1) and the inequality (2.2). Then

$$\begin{aligned} \|A(x) - B(x)\|_Y &= \frac{1}{2^{2n}} \|A(2^n x) - B(2^n x)\|_Y \\ &\leq \frac{1}{2^{2n}} \left(\|A(2^n x) - g(2^n x)\|_Y + \|g(2^n x) - B(2^n x)\|_Y \right) \\ &\leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{\infty} \frac{1}{2^{(2-s)(k+m)}} \|x\|_X^s \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for all $x \in X$. Therefore A is unique. This completes the proof of the theorem.

Theorem 2.2 Let λ and $s(s > 2)$ be non-negative real numbers. Let $g: X \rightarrow Y$ be a mapping fulfilling

$$\begin{aligned} \left\| g\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n g\left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) - (n-3) \sum_{1 \leq i < j \leq n} g(x_i + x_j) - (-n^2 + 5n - 2) \sum_{i=1}^n g(x_i) \right\|_Y \\ \leq \lambda \left\{ \|x_1\|_X^s + \|x_2\|_X^s + \dots + \|x_n\|_X^s \right\} \end{aligned} \quad (2.8)$$

for all $x_1, x_2, \dots, x_n \in X$ with $x_1 \perp x_2 \perp \dots \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A: X \rightarrow Y$ such that

$$\|g(x) - A(x)\|_Y \leq \frac{\lambda \|x\|_X^s}{2(n-5)(2^s - 2^2)} \quad (2.9)$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{2}$ in (2.3), we arrive

$$\left\| 4g\left(\frac{x}{2}\right) - g(x) \right\|_Y \leq \frac{1}{2(n-5)2^s} \lambda \left\{ \|x\|_X^s \right\} \quad (2.10)$$

for all $x \in X$. Now replacing x by $\frac{x}{2}$ and multiply by 4 in (2.10), we obtain

$$\left\| 4g\left(\frac{x}{2}\right) - 2^4 g\left(\frac{x}{2^2}\right) \right\|_Y \leq \frac{\lambda}{2(n-5)2^{2(s-1)}} \|x\|_X^s \quad (2.11)$$

From (2.10) and (2.11), we get

$$\left\| g(x) - 2^4 g\left(\frac{x}{2^2}\right) \right\|_Y \leq \frac{\lambda}{2(n-5)2^s} \left\{ 1 + \frac{1}{2^{s-2}} \right\} \|x\|_X^s \quad (2.12)$$

for all $x \in X$. Using the induction on n , we receive

$$\left\| 2^{2n} g\left(\frac{x}{2^n}\right) - g(x) \right\|_Y \leq \frac{\lambda}{2(n-5)2^s} \sum_{k=0}^{n-1} \frac{2^{2k}}{2^{sk}} \|x\|_X^s \leq \frac{\lambda}{2(n-5)2^s} \sum_{k=0}^{\infty} \frac{2^{2k}}{2^{sk}} \|x\|_X^s \quad (2.13)$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ 2^{2n} g\left(\frac{x}{2^n}\right) \right\}$ replace x by $\frac{x}{2^m}$ and multiply by 2^{2m} in (2.13), for any $n, m > 0$ we get

$$\begin{aligned} \left\| 2^{2(n+m)} g\left(\frac{x}{2^{n+m}}\right) - 2^{2m} g\left(\frac{x}{2^m}\right) \right\|_Y &= 2^{2m} \left\| 2^{2n} g\left(\frac{x}{2^{n+m}}\right) - g\left(\frac{x}{2^m}\right) \right\|_Y \\ &\leq \frac{2^{2m} \lambda}{2(n-5)2^s} \sum_{k=0}^{n-1} \frac{2^{2k}}{2^{sk}} \left\| \frac{x}{2^m} \right\|_X^s \end{aligned} \quad (2.14)$$

As $s > 2$, the R.H.S. of (2.14) tends to 0 as $m \rightarrow \infty$ for all $x \in X$. Thus $\left\{ 2^{2n} g\left(\frac{x}{2^n}\right) \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X \rightarrow Y$ such that

$$A(x) = \lim_{n \rightarrow \infty} 2^{2n} g\left(\frac{x}{2^n}\right), \quad \forall x \in X.$$

Taking $n \rightarrow \infty$ in (2.13), we have at the formula (2.9) for all $x \in X$. To prove the uniqueness of A and it satisfies the equation (1.1), the proof is similar to that of Theorem 2.1

3. Stability of the Orthogonally Euler-Lagrange Type Functional Equation (1.1) involving Constant only.

Theorem 3.1 Let λ be non-negative real number. Let $g: X \rightarrow Y$ be a mapping fulfilling

$$\left\| g\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n g\left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) - (n-3) \sum_{1 \leq i < j \leq n} g(x_i + x_j) - (-n^2 + 5n - 2) \sum_{i=1}^n g(x_i) \right\|_Y \leq \lambda \quad (3.1)$$

for all $x_1, x_2, \dots, x_n \in X$ with $x_1 \perp x_2 \perp \dots \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A: X \rightarrow Y$ such that

$$\|g(x) - A(x)\|_Y \leq \frac{\lambda}{6(n-5)} \quad (3.2)$$

for all $x \in X$.

Proof. Setting $(x_1, x_2, \dots, x_n)$ by $(0, 0, \dots, 0)$ in (3.1), we have $f(0) = 0$. Replacing $(x_1, x_2, \dots, x_n)$ by $(x, -x, x, -x, x, 0, \dots, 0)$ in (3.1), we obtain

$$\|2(n-5)g(2x) - 8(n-5)g(x)\|_Y \leq \lambda \quad (3.3)$$

for all $x \in X$. Since $x \perp 0$, we get

$$\left\| \frac{g(2x)}{4} - g(x) \right\|_Y \leq \frac{1}{8(n-5)} \lambda \quad (3.4)$$

for all $x \in X$. Now replacing x by $2x$ and divided by 4 in (3.4) and summing the resulting inequality with (3.4), we obtain

$$\left\| \frac{g(2^2 x)}{2^4} - g(x) \right\|_Y \leq \frac{\lambda}{8(n-5)} \left\{ 1 + \frac{1}{2^2} \right\} \quad (3.5)$$

for all $x \in X$. Using the induction on n , we receive

$$\left\| \frac{g(2^n x)}{2^{2n}} - g(x) \right\|_Y \leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{n-1} \frac{1}{2^{2k}} \leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \quad (3.6)$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \frac{g(2^n x)}{2^{2n}} \right\}$ replace x by $2^m x$

and divided by 2^{2m} in (3.6), for any $n, m > 0$ we get

$$\begin{aligned} \left\| \frac{g(2^{n+m} x)}{2^{2(n+m)}} - \frac{g(2^m x)}{2^{2m}} \right\|_Y &= \frac{1}{2^{2m}} \left\| \frac{g(2^{n+m} x)}{2^{2n}} - g(2^m x) \right\|_Y \\ &\leq \frac{\lambda}{8(n-5)2^{2m}} \sum_{k=0}^{n-1} \frac{1}{2^{2k}} \\ &\leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{\infty} \frac{1}{2^{2(k+m)}} \end{aligned} \quad (3.7)$$

As $s < 2$, the R.H.S. of (3.7) tends to 0 as $m \rightarrow \infty$ for all $x \in X$. Thus $\left\{ \frac{g(2^n x)}{2^{2n}} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X \rightarrow Y$ such that

$$A(x) = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{2^{2n}}, \quad \forall x \in X.$$

Taking $n \rightarrow \infty$ in (3.6), we have at the formula (3.2) for all $x \in X$. To prove A satisfies (1.1), replace $(x_1, x_2, \dots, x_n)$ by $(2^n x_1, 2^n x_2, \dots, 2^n x_n)$ in (3.1) and divided by 2^{2n} then it follows that

$$\frac{1}{2^{2n}} \left\| g \left(\sum_{i=1}^n 2^n x_i \right) + \sum_{j=1}^n g \left(2^n \left(-x_j + \sum_{i=1, i \neq j}^n x_i \right) \right) - (n-3) \sum_{1 \leq i < j \leq n} g \left(2^n (x_i + x_j) \right) - (-n^2 + 5n - 2) \sum_{i=1}^n g \left(2^n x_i \right) \right\|_Y \leq \frac{\lambda}{2^{2n}}$$

Again taking limit as $n \rightarrow \infty$ in the above inequality, we have

$$\left\| A \left(\sum_{i=1}^n x_i \right) + \sum_{j=1}^n A \left(-x_j + \sum_{i=1, i \neq j}^n x_i \right) - (n-3) \sum_{1 \leq i < j \leq n} A(x_i + x_j) - (-n^2 + 5n - 2) \sum_{i=1}^n A(x_i) \right\|_Y = 0$$

which gives

$$g \left(\sum_{i=1}^n x_i \right) + \sum_{j=1}^n g \left(-x_j + \sum_{i=1, i \neq j}^n x_i \right) = (n-3) \sum_{1 \leq i < j \leq n} g(x_i + x_j) + (-n^2 + 5n - 2) \sum_{i=1}^n g(x_i)$$

for all $x_1, x_2, \dots, x_n \in X$ with $x_1 \perp x_2 \perp \dots \perp x_n$. Therefore $A: X \rightarrow Y$ is an orthogonally quadratic mapping which satisfies (1.1).

To prove the uniqueness of A. Let B be another quadratic function satisfying (1.1) and the inequality (3.2). Then

$$\begin{aligned} \|A(x) - B(x)\|_Y &= \frac{1}{2^{2n}} \|A(2^n x) - B(2^n x)\|_Y \\ &\leq \frac{1}{2^{2n}} \left(\|A(2^n x) - g(2^n x)\|_Y + \|g(2^n x) - B(2^n x)\|_Y \right) \\ &\leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{\infty} \frac{1}{2^{(2-s)(k+m)}} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for all $x \in X$. Therefore A is unique. This completes the proof of the theorem.

Theorem 3.2 Let λ be non-negative real numbers. Let $g: X \rightarrow Y$ be a mapping fulfilling

$$\left\| g \left(\sum_{i=1}^n x_i \right) + \sum_{j=1}^n g \left(-x_j + \sum_{i=1, i \neq j}^n x_i \right) - (n-3) \sum_{1 \leq i < j \leq n} g(x_i + x_j) - (-n^2 + 5n - 2) \sum_{i=1}^n g(x_i) \right\|_Y \leq \lambda \quad (3.8)$$

for all $x_1, x_2, \dots, x_n \in X$ with $x_1 \perp x_2 \perp \dots \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A: X \rightarrow Y$ such that

$$\|g(x) - A(x)\|_Y \leq \frac{\lambda}{-6(n-5)} \quad (3.9)$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{2}$ in (3.3), we arrive

$$\left\| 4g\left(\frac{x}{2}\right) - g(x) \right\|_Y \leq \frac{1}{2(n-5)} \lambda \quad (3.10)$$

for all $x \in X$. Now replacing x by $\frac{x}{2}$ and multiply by 4 in (3.10), we obtain

$$\left\| 4g\left(\frac{x}{2}\right) - 2^4 g\left(\frac{x}{2^2}\right) \right\|_Y \leq \frac{2^2 \lambda}{2(n-5)} \quad (3.11)$$

From (3.10) and (3.11), we get

$$\left\| g(x) - 2^4 g\left(\frac{x}{2^2}\right) \right\|_Y \leq \frac{\lambda}{2(n-5)} \left\{ 1 + \frac{1}{2^{-2}} \right\} \quad (3.12)$$

for all $x \in X$. Using the induction on n , we receive

$$\left\| 2^{2n} g\left(\frac{x}{2^n}\right) - g(x) \right\|_Y \leq \frac{\lambda}{2(n-5)} \sum_{k=0}^{n-1} 2^{2k} \leq \frac{\lambda}{2(n-5)} \sum_{k=0}^{\infty} 2^{2k} \quad (3.13)$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ 2^{2n} g\left(\frac{x}{2^n}\right) \right\}$ replace x by $\frac{x}{2^m}$

and multiply by 2^{2m} in (3.13), for any $n, m > 0$ we get

$$\begin{aligned} \left\| 2^{2(n+m)} g\left(\frac{x}{2^{n+m}}\right) - 2^{2m} g\left(\frac{x}{2^m}\right) \right\|_Y &= 2^{2m} \left\| 2^{2n} g\left(\frac{x}{2^{n+m}}\right) - g\left(\frac{x}{2^m}\right) \right\|_Y \\ &\leq \frac{2^{2m} \lambda}{2(n-5)} \sum_{k=0}^{n-1} 2^{2k} \end{aligned} \quad (3.14)$$

As $s > 2$, the R.H.S. of (3.14) tends to 0 as $m \rightarrow \infty$ for all $x \in X$. Thus $\left\{ 2^{2n} g\left(\frac{x}{2^n}\right) \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X \rightarrow Y$ such that

$$A(x) = \lim_{n \rightarrow \infty} 2^{2n} g\left(\frac{x}{2^n}\right), \forall x \in X.$$

Taking $n \rightarrow \infty$ in (3.13), we have at the formula (3.9) for all $x \in X$. To prove the uniqueness of A and it satisfies the equation (1.1), the proof is similar to that of Theorem 3.1

4. Stability of the Orthogonally Euler-Lagrange Type Functional Equation (1.1) involving Sum of the Product of Powers of Norms

Theorem 4.1 Let λ and $s\left(s < \frac{2}{n}\right)$ be non-negative real numbers. Let $g: X \rightarrow Y$ be a mapping fulfilling

$$\begin{aligned} & \left\| g\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n g\left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) - (n-3) \sum_{1 \leq i < j \leq n} g(x_i + x_j) - (-n^2 + 5n - 2) \sum_{i=1}^n g(x_i) \right\|_Y \\ & \leq \lambda \left\{ \left(\|x_1\|_X^s \|x_2\|_X^s \dots \|x_n\|_X^s \right) + \left(\|x_1\|_X^{ns} + \|x_2\|_X^{ns} + \dots + \|x_n\|_X^{ns} \right) \right\} \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in X$ with $x_1 \perp x_2 \perp \dots \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A: X \rightarrow Y$ such that

$$\|g(x) - A(x)\|_Y \leq \frac{\lambda \|x\|_X^{ns}}{2(n-5)(2^2 - 2^{ns})}$$

for all $x \in X$.

Proof. The proof is same as the proof of Theorem 2.1.

Theorem 4.2 Let λ and $s\left(s > \frac{2}{n}\right)$ be non-negative real numbers. Let $g: X \rightarrow Y$ be a mapping fulfilling

$$\begin{aligned} & \left\| g\left(\sum_{i=1}^n x_i\right) + \sum_{j=1}^n g\left(-x_j + \sum_{i=1, i \neq j}^n x_i\right) - (n-3) \sum_{1 \leq i < j \leq n} g(x_i + x_j) - (-n^2 + 5n - 2) \sum_{i=1}^n g(x_i) \right\|_Y \\ & \leq \lambda \left\{ \left(\|x_1\|_X^s \|x_2\|_X^s \dots \|x_n\|_X^s \right) + \left(\|x_1\|_X^{ns} + \|x_2\|_X^{ns} + \dots + \|x_n\|_X^{ns} \right) \right\} \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in X$ with $x_1 \perp x_2 \perp \dots \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A: X \rightarrow Y$ such that

$$\|g(x) - A(x)\|_Y \leq \frac{\lambda \|x\|_X^{ns}}{2(n-5)(2^{ns} - 2^2)}$$

for all $x \in X$.

Proof. The proof is same as the proof of Theorem 2.2.

References

- [1] J. Alonso and C. Benitez, Orthogonality in normed linear spaces: survey I. main properties, Extracta Math., Vol. 3, (1988), 1-15.
- [2] J. Alonso and C. Benitez, Orthogonality in normed linear spaces: a survey II. Relations between main orthogonalities, Extracta Math., Vol. 4, (1989), 121-131.
- [3] F. Drjevic, On a functional which is quadratic on a orthogonal vectors, Publ. Inst. Math., Vol. 54, (1986), 63-71.
- [4] M. Fochi, Functional equations in a orthogonal vectors, Aequationes Math., Vol. 38, (1989), 28-40.
- [5] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., Vol. 184, (1994), 431-436.
- [6] R. Ger and J. Sikorska, Stability of the orthogonal additivity, Bull. Polish Acad. Sci. Math., Vol. 43, (1995), 143-151.
- [7] V. Govindan, S. Murthy, M. Saravanan, Generalized Ulam-Hyers Stability of two types of n-dimensional Quadratic functional equation in Banach Space: Direct and Fixed Point Methods. Int. J. Math Trends and its Tech., Vol. 51(6), (2017), 396-403
- [8] V. Govindan, S. Murthy, G. Kokila, Fixed point and stability of icosic functional equation in quasi beta normed spaces, Malaya J. mathematic, Vol. 6(1), (2018), 261-275
- [9] V. Govindan and K. Tamilvanan, Stability of Functional Equation in Banach Space: Using Two Different Methods, Int. J. Math. And Appl., Vol. 6(1-C), (2018), 527-536.
- [10] S. Gudder and D. Strawther, Orthogonally additive and orthogonally increasing functions on vector spaces, Pacific J. Math. Vol. 58, (1995), 427-436.
- [11] L. Paganoni and J. Ratz, Conditional functional equations and orthogonal additivity, Aequationes math., Vol. 50, (1995), 135-142.
- [12] C. Park and Th. M. Rassias, Fixed points and stability of the Cauchy functional equation, Australian J. Math. Anal. Appl., Vol. 6(1), (2009), 1-9.
- [13] J. M. Rassias, On approximately of approximately linear mappings by linear mappings, J. Funct. Anal., USA, Vol. 46, (1982), 126-130.
- [14] J. M. Rassias, On the stability of the non-linear Euler-Lagrange functional equation in real normed linear spaces, J. Math. Phys. Sci., Vol. 28, (1994), 231-235.
- [15] J. M. Rassias, On the stability of the general Euler-Lagrange functional equation, Demonstratio Math., Vol. 29, (1996), 755-766.
- [16] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., Vol. 72, (1978), 297-300.

How do I cite this article?

M.Settu et., al, ON THE ULAM-GAVRUTA-RASSIAS STABILITY OF THE ORTHOGONALLY EULER-LAGRANGE TYPE FUNCTIONAL EQUATION, Excel International Journal of Technology, Engineering and Management, 2026; Volume -13, Issue-4_Page_21-31.



This is an open access article under the CC BY-NC-ND license

(<http://creativecommons.org/licenses/by-nc-nd/4.0/>)